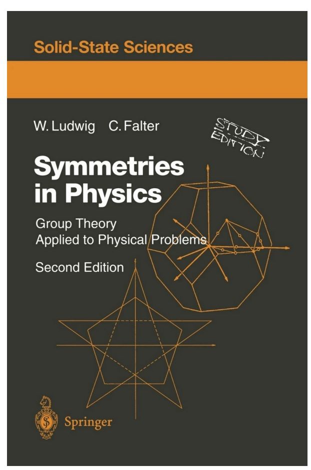
Irrational near-groups

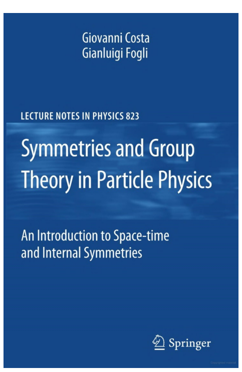
ANDREW SCHOPIERAY

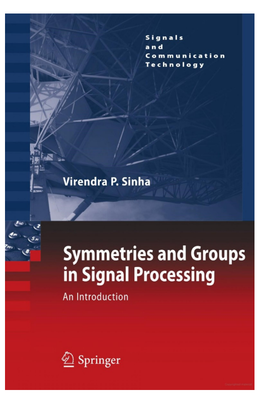
University of Alberta
&
Pacific Institute for the Mathematical Sciences

Finite groups act on various things

This is widely regarded as useful





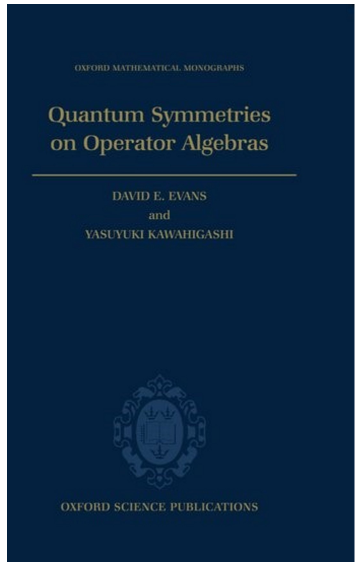


Fusion categories act on various things

We are still learning how useful this is

Lecture: Various things acted on by fusion categories

André Henriques



Fusion

Fusion Ring: (R,B)

Associative, unital ring R, free as a $\mathbb{Z}$-module, with basis $B = \{1_R = b_0, b_1, \dots, b_n\}$ with

- **(POSITIVITY)** $b_i b_j = \sum_{k=0}^{n} c_{i,j}^{k} b_k$, then $c_{i,j}^{k} \in \mathbb{Z}_{\geq 0}$
- **(DUALITY)** there exists an involution of B, $x \mapsto x^*$ such that $c_{i,j}^{0} = \delta_{i,j^*}$

E.g. Integral group rings

The most elementary of all fusion matrices are permutation matrices

$$\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix} \begin{bmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\end{bmatrix} \begin{bmatrix}0&0&1&0&0&0\\0&0&0&0&1&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&1&0&0&0&0\\0&0&0&1&0&0\end{bmatrix} \begin{bmatrix}0&0&0&1&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\\1&0&0&0&0&0\\0&0&1&0&0&0\\0&1&0&0&0&0\end{bmatrix} \begin{bmatrix}0&0&0&0&0&1\\0&0&0&1&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\1&0&0&0&0&0\\0&0&0&0&1&0\end{bmatrix} \begin{bmatrix}0&0&0&0&1&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&1&0&0&0&0\\0&0&0&0&0&1\\1&0&0&0&0&0\end{bmatrix}$$

Fig. 0 $\mathbb{Z}S_3$

If all fusion matrices of elements of B are permutations, then B has the structure of a finite group G

These fusion rings are precisely $\mathbb{Z}G$ for finite groups G

E.g. Near-group rings

Assume there exists a unique basis element ρ whose fusion matrix is not a permutation

Then $B - \{\rho\}$ has the structure of a finite group G

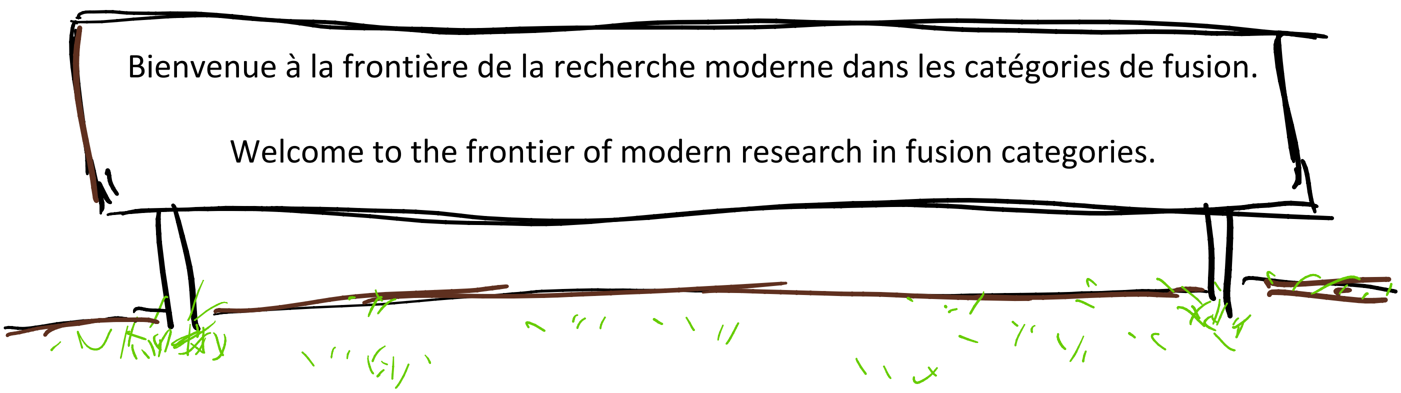
And the fusion of R is determined by $c_{\rho,\rho}^{\rho} := \ell$.

$$\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\1&1&1&1&1&\ell\end{bmatrix}$$

Fig. 1 Fusion matrix of ρ in $R(C_5, \ell)$

So we may denote the near-group fusion rings by $R(G, \ell)$

For specific examples, the character ring of S_3 is $R(C_2, 1)$ and the character ring of A_4 is $R(C_3, 2)$.


Bienvenue à la frontière de la recherche moderne dans les catégories de fusion.
Welcome to the frontier of modern research in fusion categories.

Categories

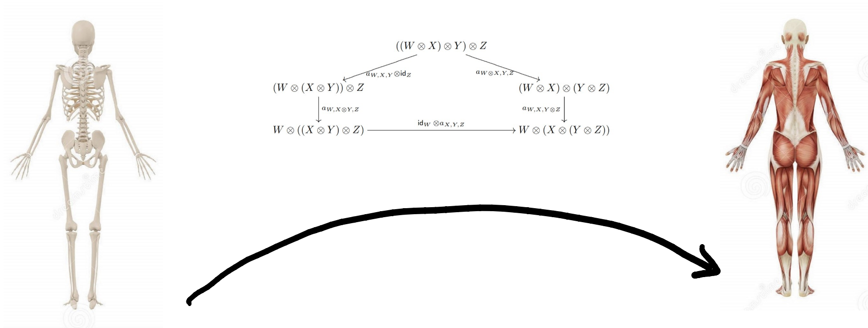


Fig. 2a Fusion Ring

Fig. 2b Fusion category

Fusion categories are fusion rings with a cohesive collection of associativity data (i.e. 6j-symbols, F-matrices, solutions to pentagons, etc.)

E.g. Finite groups as fusion categories

The categorifications of the integral groups rings (over $\mathbb{C}$, up to equivalence) are the fusion categories of G-graded complex vector spaces…

$$\omega(g_1g_2, g_3, g_4)\omega(g_1, g_2, g_3g_4) = \omega(g_1, g_2, g_3)\omega(g_1, g_2g_3, g_4)\omega(g_2, g_3, g_4)$$

Fig. 3 Pentagon equations for Vec_G^{ω}

…with associativity defined by $\omega \in H^3(G, \mathbb{C}^{\times})$

But essentially no fusion rings are categorifiable

There are 161 fusion rings of rank 3 and multiplicity ≤ 16

Five of the rank 3 fusion rings are categorifiable

Rank

Multiplicity	1	2	3	4	5	6	7	8	9
1	1	2	4	10	16	39	43	96	142
2	0	1	3	17	37	154	319	874+	
3	0	1	4	24	82	384	562+		
4	0	1	6	45	134	872	1236+		
5	0	1	5	55	209	533+			
6	0	1	9	81	336	872+			
7	0	1	6	92	477	976+			
8	0	1	10	137	733	1672+			
9	0	1	12	151	1463				
10	0	1	9	186	1794				
11	0	1	10	238	2283				
12	0	1	20	291	3049				
13	0	1	9	246	1300+				
14	0	1	13	340	1323+				
15	0	1	16	349	1550+				
16	0	1	25	525	1925+				

On low rank fusion rings

Gert Vercleyen, Joost Slingerland

There is a widespread belief that for a fixed rank, there are finitely-many fusion categories up to equivalence

This would imply Level Bounds™ for categorifiable near-group fusion rings:

Conjecture: Let G be a finite group. There exists $N_G \in \mathbb{Z}_{\geq 0}$ such that $R(G, \ell)$ is not categorifiable for all $\ell \geq N_G$.

Easily proven true for nonabelian groups G.

- If FPdim($R(G, \ell)$) is an integer, then $\ell < |G|$.
- If $R(G, \ell)$ is categorifiable and FPdim($R(G, \ell)$) is **irrational**, then G is abelian. (and $\ell = k|G|$)

ALGEBRAIC REALIZATION OF NONCOMMUTATIVE NEAR-GROUP FUSION CATEGORIES

Masaki Izumi & Henry Tucker

Proven ``incidentally'' for three abelian groups G.

- ($G = C_1$) Victor Ostrik classified all rank 2 fusion categories [20 years ago].
- ($G = C_2$) Victor Ostrik classified all (pivotal) rank 3 fusion categories [10 years ago].
- ($G = C_3$) Hannah Larson classified all pseudounitary non-self dual rank 4 fusion categories [8 years ago].

Open problem: Let G be a finite abelian group. Is the collection of $k \in \mathbb{Z}_{\geq 0}$ such that $R(G, k|G|)$ is categorifiable a finite set?

Tensor Categories with Fusion Rules of Self-Duality for Finite Abelian Groups

Daisuke Tambara & Shigeru Yamagami

A Cuntz algebra approach to the classification of near-group categories

Masaki Izumi

Near-group fusion categories and their doubles

David Evans & Terry Gannon

PhD thesis of Paul Budinski

Theorem (Schopieray, 2022): Let G be an elementary abelian 2-group. Then $R(G, \ell)$ is categorifiable if and only if

- $\ell = 0$;

  (categorification: 20+ years ago)

- $G = C_2$ and $\ell = 1$ or $\ell = 2$;

  (categorification: 25+ years ago)

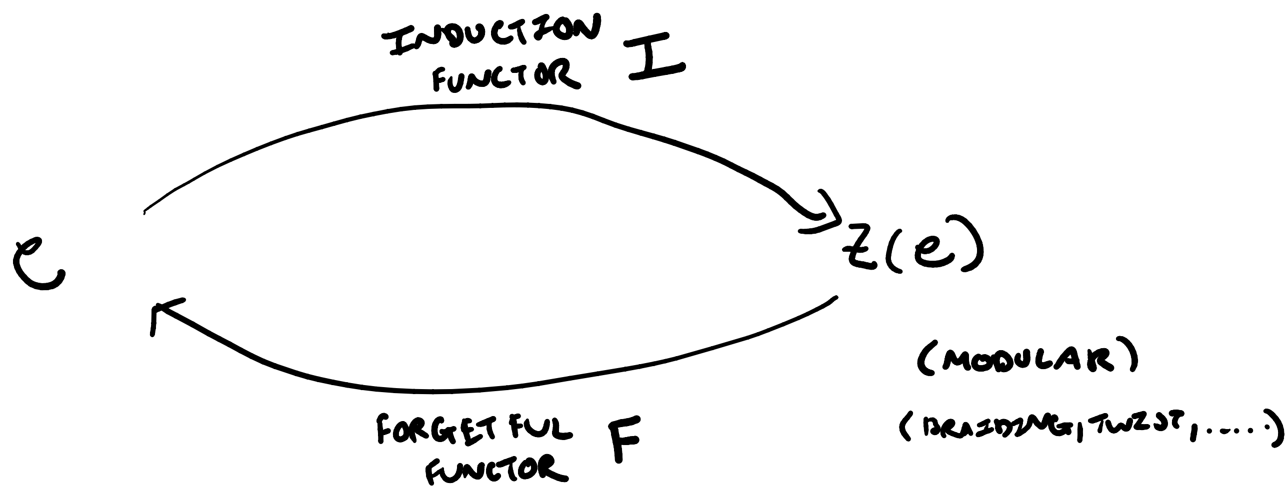
- $G = C_2^2$ and $\ell = 4$.

  (categorification: 20+ years ago)

Contained in: Categorification of integral group rings extended by one dimension

Proof outline:

Induction & restriction:



Motivation: $Z(\mathrm{Vec}_G^\omega)$ ARE COMPLETELY UNDERSTOOD

IN PARTICULAR, $I(g)$ FOR INVERTIBLE OBJECTS.

Observation: IF $\mathcal{C}$ IS A NEAR-GROUP FUSION CATEGORY,

$I(g)$, $g \in G$ ARE "UNCHANGED" FOR $k \in \mathbb{Z}_{\geq 0}$

↑ ACCOUNTS FOR

$\frac{1}{2}|G|(|G|+3)$ SIMPLE OBJECTS IN $Z(\mathcal{C})$

Consequence: IF k IS UNBOUNDED, $I(\rho)$ MUST BE "INFINITELY INTERESTING".

Spoiler: IT'S NOT. (FROM A NUMBER-THEORETIC PERSPECTIVE) //

E.g. The above argument fails to prove finiteness when $G = C_7$ (for example)

But a computer check of the constraints for low levels implies

$R(C_7, 7)$,	$R(C_7, 42)$,	$R(C_7, 70)$,	$R(C_7, 672)$
$R(C_7, 10\,710)$,	$R(C_7, 49\,210)$,	$R(C_7, 170\,688)$	
$R(C_7, 2\,720\,298)$			

are the only possible categorifiable $R(C_7, \ell)$ with $\ell < 14\,000\,000$

A vast generalization to this finiteness:

If (R, B) is a fusion ring, and there exists $d \notin \mathbb{Z}$ such that $\mathrm{FPdim}(b) \in \{1, d\}$ for all $b \in B$, then the set of invertible basis elements G acts transitively on $B - G$.

The stabilizer subgroups of $b \in B - G$ are the same (normal) subgroup $H \leq G$.

Integral counterexamples: character rings of extraspecial p-groups, $p > 2$

These include irrational near-group rings, Haagerup-Izumi rings, ``quadratic" rings, and much much more!

Lemma (Schopieray, 2022): If such a fusion ring is categorifiable, then d is a root of $x^2 - k|H|x - |H|$ for some $k \in \mathbb{Z}_{\geq 0}$.

Note: The case $G = H$ is the well-known result for near-group fusion categories

The Frobenius-Perron dimension of the category in this case is $|G|(2 + kd)$.

The global dimension of such a fusion category is $|G|(2 + kd)$ or its Galois conjugate.

PhD thesis of Josiah Thornton

Conjecture: Let $N \in \mathbb{Z}_{\geq 1}$. Then there exists $\delta_N > 0$ such that for all fusion categories $\mathcal{C}$ with N invertible objects up to isomorphism, $\dim(\mathcal{C}) > N + \delta_N$ or $\dim(\mathcal{C}) = \mathrm{N}$.

Note: the case $N = 1$ was proven by Victor Ostrik in 2018. ($\delta_1 = 1/3$ suffices)

Proof in the modular case: $\dim(\mathcal{C}) = \mathrm{N} \cdot \dim(\mathcal{C}_{ad}) > \frac{4}{3} \cdot \mathrm{N}$

Proof in the braided case: Exercise.

Proof in general: Let me know when you've proven it.

For a fixed $r \in \mathbb{Z}_{\geq 1}$, there exist finitely-many categorifiable fusion rings of rank r whose basis elements take exactly one nontrivial Frobenius-Perron dimension d.

Rank-finiteness is known for integral fusion categories.

So assume $d \notin \mathbb{Z}$.

There are finitely-many possible G since $|G| < r$. Fix G.

Now d is a root of $x^2 - k|H| - |H|$ for some $k \in \mathbb{Z}_{\geq 0}$ and subgroup $H \leq G$.

Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be a Galois automorphism with $\sigma(d) \neq d$.
We may assume $\dim(\mathcal{C}) = |G|(2 + kd)$ since $\mathcal{C}$ is Galois conjugate to a pseudounitary fusion category.

Then

$$\lim_{k\to\infty} \mathcal{C}^\sigma = \lim_{k\to\infty} \sigma(\dim(\mathcal{C})) = \lim_{k\to\infty} |G|(2 + k\sigma(d)) = |G|.$$

Therefore, (**conjecturally**) k is bounded above for a fixed G.

Moreover, there are only finitely-many categorifiable fusion rings for fixed G and k.

Thank you for your attention!